COMBINATORICA
Akadémiai Kiadó – Springer-Verlag

COMBINATORICA 11 (2) (1991) 157–171

ON THE POWER OF WHITE PEBBLES[1]

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Received February 25, 1988
Revised May 22, 1989

We construct a family $(G_p \mid p)$ of directed acyclic graphs such that any black pebble strategy for G_p requires p^2 pebbles whereas $3p+1$ pebbles are sufficient when white pebbles are allowed.

1. Introduction

The black pebble game is played on a directed acyclic graph G of bounded indegree. The game is played by placing and removing pebbles on the vertices of G. We describe the rules of the game.

(1) A black pebble may be placed on a vertex v if and only if all immediate predecessors of v have pebbles.

(2) A black pebble can be removed at any time.

Starting with a pebble-free graph, the goal is to pebble all vertices at some time. A sequence of moves achieving this goal is called a strategy. The number of pebbles used by a strategy is the maximum over all time steps t of the number of pebbles on the graph at time t.

The black pebble game was introduced by Hewitt and Paterson [2] to model the deterministic evaluation of straight line programs. The minimal number of pebbles required to pebble the "flow chart" of the program corresponds to the deterministic space required for evaluation.

Cook and Sethi [1] introduced the black-white pebble game to investigate the nondeterministic evaluation of straight line program. White pebbles are introduced to simulate nondeterministic guesses. Correspondingly the following two rules are added to define the black-white pebble game.

(3) A white pebble can be placed on a vertex at any time.

(4) A white pebble can be removed only if all of its immediate predecessors have pebbles.

AMS subject classification (1980): 68 Q 05, 68 R 05, 68 R 10

[1] Supported by the National Science Foundation under contract no. DCR-8407256 and by the office of Naval Research under contract no. N0014-80-0517.

Now, a successful pebbling of the graph consists of pebbling each vertex at some time and finally removing all pebbles. The definition of a strategy and the number of pebbles required by a strategy are analogous to the black pebble game.

Pippenger [6] gives an extensive survey on pebble games. Meyer auf der Heide [5] showed that any graph with a black-white pebble strategy requiring p pebbles can be pebbled with $(p^2 - p)/2 + 1$ black pebbles.

For a broad class of graphs (including pyramid graphs), Klawe [3] proved that the two games can not be separated by a factor larger than 2. On the other hand, Meyer auf der Heide [5] showed that a factor of 2 is achievable for pyramid graphs.

Lengauer and Tarjan [4] consider *Time/Space* tradeoff's. For the black pebble game, they obtain the trade-off $T = \theta(N^2/S)$ for a certain family of graphs. Here N is the number of vertices, S is the number of black pebbles and T is the number of moves. The corresponding trade-off for the black-white game is $T = \theta(N^2/S^2) + \theta(N)$. Their result implies a quadratic difference between the two pebble games *provided* the number of moves is fixed.

Wilber [8] obtained the best separation result so far. He exhibited a family of graphs $(W_p \mid p)$ that can be pebbled with p black-white pebbles but require $\Omega(p \log p / \log\log p)$ black pebbles. Our result is

Theorem 1. *There is a family* $(G_p^j \mid p, j)$ *of directed acyclic graphs of indegree three such that*

(1) G_p^j *can be pebbled with* j *white pebbles and* $p + j + 1$ *black pebbles.*

(2) Any black strategy needs at least jp *black pebbles to pebble the graph.*

(3) Let v *be equal to the number of vertices of* G_p^j*. Then* $v = \theta((p+1)^{j+1})$

For the choice of $j = p$, Theorem 1 shows a quadratic difference (in the number of pebbles) between the black- and the black-white pebble game. Consequently, Meyer auf der Heide's upper bound is asymptotically tight. Observe the striking similarity between this upper bound and Savitch's theorem [7].

Let us quantify the amount of nondeterminism by the number of white pebbles used in the black-white pebble game. Then, Theorem 1 also shows that $\theta(pj)$ black pebbles are needed to compensate for the absence of j white pebbles. This result is asymptotically the best separation for the two games as shown by,

Theorem 2. *Any black-white strategy that uses at most* i *white pebbles and* p *black pebbles can be simulated by a black strategy that uses at most* $(i+1)(p+3i/2)$ *black pebbles.*

Wilber's separation result holds for graphs of size polynomial in p. Observe that Theorem 1 does not achieve this. But, with a slight modification of our techniques, we will be able to obtain Wilber's separation factor for polynomial size graphs.

Theorem 3. *There exists a family* $(H_{p,k}^j \mid j \le p/k)$ *of directed acyclic graphs of indegree three such that*

(1) $H_{p,k}^j$ *can be pebbled with* kj *white pebbles and* $p + kj + 1$ *black pebbles whereas*

(2) jp *black pebbles are necessary.*

(3) Let v *be the number of vertices in* $H_{p,k}^j$*. Then* $v = O(poly(p)(p/k)^j)$.

Now, setting $k = p \log\log(p)/\log(p)$, we obtain a family of graphs of size $poly(p)$ requiring $p\log(p)/\log\log(p)$ black pebbles whereas $O(p)$ black-white pebbles suffice. Also, (for $k = 1$) we obtain Theorem 1 as a special case of Theorem 3.

Our black-white strategies for G_p^j as well as for $H_{p,k}^j$ pebble any vertex exactly once. Results of Wilber [9] imply that $O(p\log(\text{size of the graph}))$ black pebbles suffice when each vertex is pebbled only a constant number of times in the black-white strategy. Therefore, an improvement of Theorem 3 seems to require new techniques.

First we show how to simulate white pebbles by black pebbles in section 2. In section 3, we describe the construction of the graphs G_p^j. Theorem 1 is proved in section 4. The proof of Theorem 3 is presented in section 5.

2. An Upper Bound

In this section, we show how to simulate white pebbles by black pebbles. We will simulate a strategy that uses at most i white pebbles as well as at most p black pebbles by a black-strategy that uses at most $(i+1)(p+3i/2)$ black pebbles.

We extend the black-white pebble game by introducing red pebbles. The rules for the red pebbles are identical to the rules for the black pebbles. We add the following rule to the black-white-red pebble game.

A red pebble can be converted into black pebble and a black pebble can be converted into a red pebble at any time.

Now, a successful pebbling of the graph consists of black, white or red pebbling each vertex at some time and finally removing all pebbles.

First we will convert a given black-white strategy that uses at most i white pebbles into a strategy that uses at most $i-1$ white pebbles at the cost of introducing only a few red pebbles.

This is achieved by replacing a pebble move of the given strategy that places a pebble on a vertex, say v, by a sequence of black-white-red pebble moves that places a black pebble on v while using at most $i-1$ white pebbles. During this phase some black pebbles will not be moved, since we need the original configuration for are later part of the simulation. We will convert those stationary black pebbles into red pebbles. At the end of this phase we will convert these red pebbles back into black pebbles. This conversion strategy helps us to show that the number of additional pebbles introduced during this phase only depends on the number of black and white pebbles.

First, we define some graph theoretic notions.

Definition 2.1. Let $G = (V, E)$ be a directed acyclic graph of bounded indegree.
(a) $sinks(G) = \{v \in V \mid \text{out-degree}(v) = 0\}$.
(b) Let $N \subseteq V$. We define $ancestor(N)$ to be the set of all vertices $v \in V$ such that there is a directed path from v to some vertex in N.
(c) Given $N \subseteq V$, we define the graph G_N as the subgraph of G induced by $ancestor(N)$.

Definition 2.2.
(a) We denote a configuration of black, white and red pebbles for $G = (V, E)$ by a

triple (B, W, R) of disjoint subsets of V where each vertex in B (resp. W or R) has a black (resp. a white or red) pebble on it.

(b) Let $S = [(B_i, W_i, R_i) : i = 0, \ldots, t]$ be a sequence of configurations corresponding to a sequence of valid pebble moves. We call S an (i, j, k)-strategy, provided $|B_r| \le i$, $|W_r| \le j$ and $|R_r| \le k$ for all r $(0 \le r \le t)$.

(c) We say a strategy S is monotone (i, j, k)-strategy if and only if S is an (i, j, k)-strategy with $R_s \subseteq R_{s+1}$ for $0 \le s \le t-1$.

Our first Lemma shows that under certain circumstances the number of white pebbles can be reduced by one.

Lemma 2.1. *Let Red and White be two subsets of the set of vertices of a given directed acyclic graph H, such that $|Red| \le k$ and $|White| \le i$. Let $S(White = [(\emptyset, \emptyset, Red), \ldots, (B_0, W_0 - sinks(H_{White}), R_0)]$ be a monotone $(p, i-1, k)$-strategy where $W_0 = White$ and $T(White) = [(B_r, W_r, R_r) : 0 \le r \le q]$ be a monotone (p, i, k)-strategy where $W_q = \emptyset$. Also, assume that the moves of $T(White)$ is restricted in H_{White}.*

Then, there is a sequence of monotone $(p, i-1, k+2i)$-strategies starting from $(\emptyset, \emptyset, Red)$ and ending in $(\emptyset, \emptyset, R_0 \cup W_0)$. Also, the last configuration of each monotone strategy will not contain white pebbles.

Proof. By induction on the depth d of the graph H_{White}.

Basis: $d = 0$. The claim follows trivially.

Inductive Step: Assume that the claim holds for any set U (a subset of the set of vertices of H) with depth$(H_U) < d$ and $|sinks(H_U)| \le i$. Let White be a set of vertices of H with depth$(H_{White}) = d$ and $|sinks(H_{White})| \le i$.

Let t be the first configuration in $T(White)$ without a white pebble on $sinks(H_{White})$.

Consider the induced subgraph $H_{W_t} = (V', E')$ of H. Since the pebble moves are restricted to H_{White}, W_t is a subset of the set of vertices of H_{White}. Therefore the depth of H_{W_t} is less than d. Also the number of sinks of H_{W_t} is at most $|W_t|$ $(|W_t| \le i)$.

Our first goal is to show that we can apply the claim for W_t. First we show the existence of a monotone $(p, i-1, k)$-strategy $S(W_t)$ starting from $(\emptyset, \emptyset, Red)$ and ending in $(B_t \cap V', W_t - sinks(H_{W_t}), (R_t \cap V') \cup Red)$.

We obtain a sequence $R(White)$ by first executing $S(White)$, then white-pebbling $sinks(H_{White})$ and finally executing $T(White)$. Observe that $R(White)$ is a monotone (p, i, k)-strategy. We now consider only those pebble moves in $R(White)$ that place pebbles on $V' - sinks(H_{W_t})$. A new sequence $S(W_t)$ is formed by truncating the modified sequence $R(White)$ immediately after the tth move in the sequence $T(White)$. Observe that during the first t steps of $T(White)$ at least one white pebble stays on $sink(H_{White})$. Also, sliding a pebble from a node to its ancestor is not allowed. Therefore $S(W_t)$ is an $(p, i-1, k)$-strategy. $S(W_t)$ is monotone since we are not removing any red pebbles.

We now form a strategy $T(W_t)$ starting from $(B_t \cap V', W_t, (R_t \cap V') \cup Red)$ and ending in $(B_q \cap V', \emptyset, (R_q \cap V') \cup Red)$. We obtain this strategy from $T(White)$ by deleting the first t moves and all remaining moves that do not place a pebble on V'. Observe that $T(W_t)$ is a monotone (p, i, k)-strategy.

Therefore, we can apply Lemma 2.1 for W_t to obtain a sequence of monotone $(p, i-1, k+2i)$-strategies $S_1 = [(\emptyset, \emptyset, Red), \ldots, (\emptyset, \emptyset, (R_t \cap V') \cup Red \cup W_t)]$.

Let $START = (R_t \cap V') \cup Red \cup W_t$ and $END = R_t \cup White \cup W_t$. We will show the existence of an $(p, i-1, k+p+2i)$-strategy S_2 starting from $(\emptyset, \emptyset, START)$ and ending in $(B_t, \emptyset, END)$.

First, follow the pebble moves of $S(White)$ to reach $(B_0, White - sinks(H_{White}), START \cup R_0)$ from $(\emptyset, \emptyset, START)$. We do not white pebble $sinks(H_{White})$. Now, follow the pebble moves of $T(White)$, ignoring moves that pebble already pebbled vertices. Whenever a white pebble on a vertex in $White$ is removed for the last time, replace the move by a move that red pebbles the vertex. After following the first t moves of $T(White)$, we will have the red pebbled $White$. Let S_2 be the concatenation of the two modified sequences.

Since we have at least one white pebble on $sinks(H_{White})$ in the first t moves of $T(White)$, S_2 uses at most $i-1$ white pebbles. Moreover, S_2 uses at most p black pebbles. But, additionally we introduce (at most $2i$) red pebbles for $W_0 (= White)$ and W_t. We never remove red pebbles during S_2. Therefore, S_2 is a monotone $(p, i-1, k+2i)$-strategy.

Observe that $S_1 S_2$ is a sequence of monotone $(p, i-1, k+2i)$-strategies. Now, extend $S_1 S_2$ by appending a sequence of pebble moves that removes red pebbles not in $W_0 \cup R_0$ and black pebbles from B_t. The claim follows since the appended sequence can be written as a sequence of monotone strategies with each strategy consisting of a single configuration only. ∎

The following claim will allow us to prove Theorem 2 inductively.

Lemma 2.2. *Let $G = (V, E)$ be a directed acyclic graph and let $X = [(B_r, W_r, R_r) : 0 \le r \le t]$ be a monotone (p, i, k)-strategy with $W_0 = \emptyset$, $W_t = \emptyset$ and $i \ge 1$. Then there exists a sequence $Z = [Z_1, Z_2, \ldots, Z_a]$ of strategies where*
(1) Z starts from $(B_0, \emptyset, R_0)$ and ends in $(B_t, \emptyset, R_t)$,
(2) each Z_j is a monotone $(p, i-1, k+p+3i)$-strategy which ends in a configuration without white pebbles and
(3) for each j $(1 \le j < a)$ the ending configuration of Z_j is the starting configuration of Z_{j+1}.

Proof. During the strategy X, let us assume that white pebbles are placed b times. We split the strategy X into b substrategies $[X_1, \ldots, X_b]$ where the split occurs just after the placement of a white pebble. For $1 \le i \le b$, let w_i be the only vertex that receives white pebble during X_i.

Observe that each X_m is a monotone (p, i, k)-strategy. For each such strategy, we would like to reduce the number of white pebbles by one. First, we transform each X_c into a new strategy Y_c by the following modifications. Let us assume that the white pebble placed on w_c (at the end of X_c) is removed in strategy X_d. Starting from the strategy X_{c+1} and ending in X_d, we replace the white pebble on w_c by a red pebble. Ignore the move that removes the white pebble in X_d and retain the red pebble throughout X_d. For each c $(1 \le c \le b)$, let Y_c be the modification of strategy X_c. Observe that Y_c is a monotone $(p, 1, k+i)$-strategy because at most i white pebbles are converted to red pebbles which stay throughout the strategy. But some additional work is needed since a simple concatenation of Y's will not be a valid sequence of pebble moves.

Each Y_c is cut into two sequences where the cut occurs just before the last move (the move that places a white pebble). Observe that the first piece is a sequence of black/red pebble moves. Let *Black* be the set of vertices with black pebbles at the end of the first piece. We now add to the first piece a sequence of pebble moves that convert black pebbles on *Black* into red pebbles. Let us call the new first piece *Head* and let *Red* be the set of vertices red-pebbled at the end of *Head*. Observe that *Head* is a monotone $(p, 0, k+p+i)$-strategy.

The second piece is a pebble move that places a white pebble on w_c. We now show how Lemma 2.1 can be applied to the second piece. Define $White := \{w_c\}$. Let $S(White) = [(\emptyset, \emptyset, Red)]$, a strategy with zero moves. We perform the following modifications to the monotone (p, i, k)-strategy $Y = (X_{c+1}, X_{c+2}, \ldots, X_b)$. We retain red pebbles on *Red* throughout the strategy. Except for the vertex w_c, each vertex having a white pebble in the first configuration of Y belongs to *Red*. We remove white pebbles from these vertices and ignore any pebble move concerning them. Now, we restrict the moves of Y even further to vertices in H_{w_c}. At the end, we drop all the black pebbles. As a result of these modifications to Y, we obtain the strategy $T(White)$ that starts in the configuration $(\emptyset, White, Red)$ and ends in the configuration $(\emptyset, \emptyset, Red)$. Observe that the strategy $T(White)$ is a monotone $(p, i, k+i+p)$-strategy.

Now, applying Lemma 2.1 we get a sequence of monotone $(p, i-1, k+p+3i)$-strategies that places a red pebble on w_c. A sequence of moves that convert red pebbles on *Black* to black pebbles is appended to the sequence of strategies obtained as a result of Lemma 2.1. Call the new sequence *Tail*. Form Z_c by concatenating *Head* and *Tail* and finally removing all the red pebbles that do not occur in the first configuration of Y_{c+1}. Observe that Z_c can be written as a sequence of monotone strategies. ∎

Proof of Theorem 2.

Claim: Any monotone (p, i, k)-strategy can be simulated by a black strategy that uses at most $(i+1)(p+3i/2)+k$ black pebbles.

Basis: $(i = 0)$ The Claim follows trivially.

Inductive step: Assume that the claim holds for $j \leq i-1$.

We have to simulate a monotone (p, i, k)-strategy. Apply Lemma 2.2 to the given strategy and obtain a sequence of monotone $(p, i-1, k+p+3i)$-strategies. As result of the inductive hypothesis, we obtain a black strategy that uses at most $i(p+3(i-1)/2)+k+p+3i = (i+1)(p+3i/2)+k$ black pebbles. ∎

3. The Construction of G_p^j

We will construct the graph G_p^j recursively. First we define G_p^0.

Definition 3.1.

(a) The "m-line" is a directed graph (V, E) with $V := \{i \mid 1 \leq i \leq m\}$ as its set of vertices and $E := \{(i, i+1) \mid 1 \leq i \leq m\}$ as its set of edges. We say that the singleton set $\{j\}$ is the jth column of the m-line.

(b) G_p^0 is defined as the p-line. The first row of G_p^0 as well as the last row is defined to be the set of all vertices.

We assume that the graph G_p^{j-1} has been constructed already. We also assume that the notions "first row", "last row" and "column" of G_p^{j-1} have been introduced, All graphs in our recursive construction will have exactly p columns.

We first introduce the basic building in the construction of G_p^j.

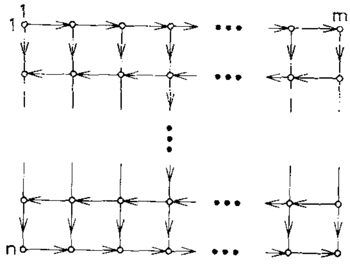


Fig. 1. The Block M_p^j

Definition 3.2. (see figure 1)

(a) The block M_p^j consists of three components, G_p^{j-1}, the $(p+1)$-line (denoted by R) and the p-line with the direction of its edges reversed. We denote the third component by A. The vertices of the last row of G_p^{j-1} are connected with the corresponding (with respect to columns) vertices of A. Also there is an edge from the sink of R to each vertex of A. Finally, there is an edge from the sink of G_p^{j-1} to the source of R.

(b) The first row of M_p^j is the first row of G_p^{j-1} and the last row of M_p^j is the set of vertices in A. The kth column of M_p^j is the kth column of G_p^{j-1} extended by the vertex in the kth column of A

We are now ready to introduce the graph G_p^j.

Definition 3.3. (see figure 2)

(a) The graph G_p^j is a sequence of $p+2$ blocks. The first block is the graph M_p^j with the vertices in R deleted.

The $(i+1)$th block $(1 \leq i \leq p)$ is an isomorphic copy of M_p^j. We connect the vertices of the last row of the ith block to the corresponding (with respect to columns) vertices of the first row of the $(i+1)$th block. Finally the vertex in the kth $(1 \leq k \leq p)$ column of the last row of the ith block is connected with the vertex in the $(p+2-k)$th column of component R of the $(i+1)$th block. All edges are directed from the ith block to the $(i+1)$th block.

The last block is an isomorphic copy of the p-line. We insert edges from the vertices of the last row of the $(p+1)$st block to the corresponding vertices of the p-line.

The first row of G_p^j is the first row of the first block. The last row of G_p^j is the last block. The kth column of G_p^j is the union of the kth columns of the $p+2$ blocks.

(b) $G_p^j(k)$ is defined to be the induced subgraph of G_p^j consisting of the first k blocks only. The first (last) row of $G_p^j(k)$ is the first (last) row of the first (kth) block of G_p^j.

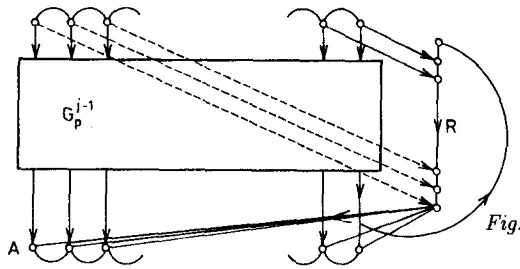


Fig. 2. The Graph G_p^j

The following Lemma summarizes the basic properties of G_p^j.

Lemma 3.1.
(a) G_p^j is a directed acyclic graph of indegree three.
(b) Let v be the number of vertices in G_p^j. Then $v = \theta((p+1)^{j+1})$.

Also, after recursively deleting all copies of R from G_p^j, the induced subgraph is a mesh with p columns and $\theta((p+1)^j)$ rows. In odd-numbered rows edges go from left to right. In even-numbered rows the directions are reversed.

One can also prove Theorem 1 for a variation of G_p^j where the direction of edges is always from "left" to "right". This would eliminate the need for the $(p+2)$nd block. But, in the case of Theorem 3, we make use of the alternating directions. Thus, our construction allows for a uniform representation.

Lemma 3.2. *There is a black-white strategy using j white pebbles and $p+j+1$ black pebbles to pebble G_p^j.*

Proof. (See figure 2.) We prove the claim inductively (on j) with the following hypothesis.

Hypothesis: Assume that each vertex in the first row of G_p^j has a black pebble. Then, there is a black-white strategy for G_p^j which results in black pebbles on each vertex of the last row. The black-white strategy uses j white pebbles and $p+j+1$ black pebbles. Also, the black-white strategy pebbles each vertex only once.

Basis: $j = 0$. Obvious because there is only one row in G_p^0.

Inductive step: Assume that the claim holds for $r = j-1$. We will prove the claim for $r = j$. First, by induction hypothesis, pebble the first block (the graph G_p^{j-1}) of G_p^j and let p black pebbles stay in the last row of the first block. So far we used only $j-1$ white pebbles and $p+(j-1)+1$ black pebbles.

Consider the second block of G_p^j. We place a white pebble on the source of R. Now using two more black pebbles, pebble the sink R. Next we advance the p pebbles from the last row of the first block to the first row of the second block.

By induction hypothesis, we continue by pebbling the second block, always retaining the white pebble placed on the source of R and the black pebble placed on the sink of R. As a consequence, the last but one row of the second block can be pebbled using $j-1$ additional white pebbles and $p+j$ additional black pebbles. Again, p black pebbles will remain on this (last but one) row. The white pebble can now be removed because the sink of G_p^{j-1} has a black pebble on it. All the p vertices in A can be pebbled and then the black pebble sitting on the sink of R can be removed. This completes the pebbling of the second block. Observe that each vertex is pebbled only once.

We repeat this process for all remaining blocks. Notice that we never used more than j white pebbles and $p + j + 1$ black pebbles. ∎

4. The Lower Bound

From now on we only consider the black pebble game. We will prove Theorem 1 by giving a lower bound on the number of black pebbles required to pebble $G_p^j(k)$ (see Definition 3.3b).

Lemma 4.1. *Suppose $j \geq 1$ and $1 \leq k \leq p+1$. Consider any configuration of black pebbles on the graph $G_p^j(k)$ in which column c is pebble-free. Let v be the vertex belonging to the last row of $G_p^j(k)$ and to column c.*
Then, any black pebble strategy to pebble v requires at least $(j-1)p + k$ pebbles.

Proof. We prove the claim by induction on j ($j \geq 1$) and k ($1 \leq k \leq p+1$)

Basis: ($j = 1$ and $k = 1$) The claim follows trivially.

First inductive step: For integers r and s, assume that the claim holds for $0 < r < j$ and for $r = j$ and $1 \leq s \leq k < p+1$. We will prove that the claim also holds for $r = j$ and $s = k+1$.

We define $Box(k+1)$ as the set of all vertices of the last block of $G_p^j(k+1)$. Let $v_0(c)$ be the vertex that belongs to the first row of $G_p^j(k+1)$ and to column c. Also, let $v_1(c)$ be the vertex belonging to the last row of the kth block of $G_p^j(k+1)$ and to column c.

We consider the time interval I that starts when a pebble is placed on $v_0(c)$ for the first time and ends when a pebble is placed on $v_1(c)$ for the first time.

Case 1: Throughout interval I, there is always a pebble in $Box(k+1)$.

With our induction hypothesis we infer that $(j-1)p+k+1$ pebbles are required.

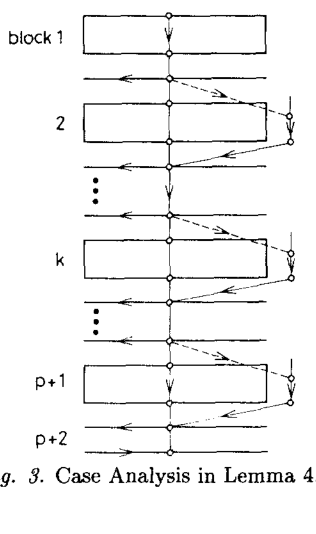


Fig. 3. Case Analysis in Lemma 4.1

Case 2: Case 1 is false (see figure 3).

Accordingly, there is a time step in interval I in which $Box(k+1)$ is pebble-free. Let $w_1(c)$ be the vertex belonging to the first row of the $(k+1)$st block of $G_p^j(k+1)$ and to column c. Denote the last but one row of $G_p^j(k+1)$ by *lbr*. Let $w_2(c)$ be the vertex belonging to *lbr* and column c.

First, we observe that each vertex in $Box(k+1)$ that does not belong to column c' ($c' < c$) is pebble-free at the end of interval I. This follows since $v_1(c)$ has to be pebbled before we can start pebbling any of those vertices. Therefore, vertices in column c which are ancestors of $w_2(c)$ and successors of $w_1(c)$ as well as $w_1(c)$ and $w_2(c)$ still have to be pebbled after interval I. Let t_1 =max(last time step of I, the latest time before vertex v is first pebbled that $Box(k+1)$ is empty). Consider the time interval J that starts when the vertex $w_1(c)$ is pebbled for the first time after t_1 and ends when vertex $w_2(c)$ is pebbled for the first time.

Case 2.1: Throughout interval J, each column of $G_p^j(k)$ (as the induced subgraph of $G_p^j(k+1)$ after removal of the last block) possesses at least one pebble.

If $j = 1$ then the last block contains a copy of G_p^0 and therefore one pebble is needed to pebble the vertex in column c. This gives a total of $p+1$ pebbles on $G_p^1(k)$. Since $k+1 \leq p+1$ we are done.

If $j > 1$, then we can apply the induction hypothesis (within interval J and for $r = j-1$ and $s = p+1$) for the copy of G_p^{j-1} in the last block. Therefore, there is time step in which at least $p + (j-1-1)p + p + 1$ pebbles are present on $G_p^j(k+1)$. We are done since $(j-1)p + k + 1 \leq jp + 1$.

Case 2.2: Case 2.1 is false.

Therefore, there exists a time step t_2 in J at which a column (say column d) in the subgraph $G_p^j(k)$ is pebble-free. We will show the existence of a time interval I' which starts with pebbling of $v_0(d)$ for the first time after t_2 and ends with pebbling of $v_1(d)$.

Remember that sometime during the interval I, $Box(k+1)$ was empty. Therefore, at t_1, either $Box(k+1)$ is empty or each vertex in $Box(k+1)$ that does not belong to a column c' ($c' < c$) is pebble-free. This implies that no pebble can be placed on a vertex belonging to R in the $(k+1)$st block until after interval J. Observe that the sink of R has to be pebbled before v, our goal vertex, can be pebbled.

Now, in order to pebble the sink of R we must pebble the vertex w of R that is connected with $v_1(d)$. Therefore, the time interval I' exists and it starts strictly after t_2. Now, during I' the $Box(k+1)$ is never empty. This situation is analogous to Case 1. Therefore, we infer that $(j-1)p + k + 1$ pebbles are required.

Second inductive step: Assume that the claim holds for $r = j$ and $s = p+1$. We will prove that the claim also holds for $r = j+1$ and $s = 1$.

Observe that the first block of G_p^{j+1} is a copy of the graph G_p^j. Since adding one more row at the end of $G_p^j(p+1)$ does not alter the lower bound, the claim also holds for $r = j+1$ and $s = 1$. ∎

5. Size versus Separation

In this section we construct a new family $(H_{p,k} \mid p, k)$ of graphs. For a special choice of k (namely $k = p \log\log p / \log p$), these graphs will be of size polynomial in the number p of black pebbles required and will separate the black game from the black-white game by a factor of $\log p / \log\log p$.

The construction of $H^j_{p,k}$ is quite similar to the construction of G^j_p. First, we define $H^0_{p,k}$.

Definition 5.1.

(a) The "(m, n)-mesh" is a directed graph (V, E) with $V := \{(i, j) \mid 1 \le i \le m, 1 \le j \le n\}$ as its set of vertices. We define column j as the set $\{(i, j) \mid 1 \le i \le m\}$. The ith row is introduced analogously as the set $\{(i, j) \mid 1 \le j \le n\}$. We call row 1 the first row and row m the last row.
The set E of edges is defined as follows,
For $1 \le i < m$ and for $1 \le j \le n$ there is an edge from (i, j) to $(i + 1, j)$.
For odd i and for all j $(1 \le j < n)$ there is an edge from (i, j) to $(i, j + 1)$.
For even i and for all j $(1 \le j < n)$ there is an edge from $(i, j + 1)$ to (i, j).

(b) $H^0_{p,k}$ is a $(2, p)$-mesh.

As before, assume that the graph $H^{j-1}_{p,k}$ has been constructed already. Moreover, we assume again that the notions "first row", "last row" and "column" of $H^{j-1}_{p,k}$ have been introduced.

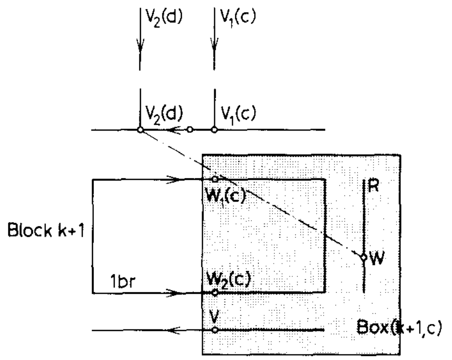


Fig. 4. The Block $N^j_{p,k}$

Definition 5.2. (See figure 4.)
This graph $N^j_{p,k}$ (which will be the building block in the construction of $H^j_{p,k}$) consists of $(k + 3)$ components,

$H^{j-1}_{p,k}$, k copies of the $(p+1)$-line (denoted by R^j_i for $1 \le i \le k$), the $(2p^2, p)$-mesh (denoted by B) and the $(4p^3 + 1, p)$-mesh called A.

The vertices of the last row of B are connected to the corresponding (with respect to columns) vertices of the first row of $H^{j-1}_{p,k}$. The vertices of the last row of $H^{j-1}_{p,k}$ are connected to the corresponding (with respect to columns) vertices of the first row of

A. For all $1 \le i \le k$, the sink of R_i^j is connected to all vertices which belong to the first row of A and to a column c with $(i-1)(p/k)+1 \le c \le i(p/k)$. The vertex in the column $(i-1)(p/k)+1$ (for $1 \le i \le k$) and the last row of $H_{p.k}^{j-1}$ is connected to the source of R_i^j. Finally, for all i $(1 \le i \le k)$, the vertex in the rth column $(1 \le r \le p)$ of the last row of B is connected with the vertex in the $(p+2-r)$th column of R_i^j.

The first row of $N_{p,k}^j$ is the first row of B and the last row of $N_{p,k}^j$ is the last row of A. The kth column of $N_{p,k}^j$ is the union of the kth column of A, B and $H_{p,k}^{j-1}$.

We now join these blocks to form the graph $H_{p,k}^j$.

Definition 5.3.

(a) The graph $H_{p,k}^j$ is a sequence of $(p/k)+1$ blocks.

The ith block $(1 \le i \le p/k+1)$ is an isomorphic copy of $N_{p,k}^j$. We connect the vertices of the last row of the ith block to the corresponding (with respect to columns) vertices of the first row of the $(i+1)$th block. All edges are directed from the ith block to the $(i+1)$th block. The first (last) row of $H_{p,k}^j$ is the first (last) row of the first (last) block.

(b) $H_{p,k}^j(m)$ is defined to be the induced subgraph of $H_{p,k}^j$ consisting of the first m blocks only. The first (last) row of $H_{p,k}^j(m)$ is the first (last) row of the first (mth) block.

Notice that the graph $H_{p,k}^j(m)$ is "almost" identical with G_p^j. The only difference is that G_p^j has copies of A that are *p-lines* whereas $H_{p,1}^j$ has copies of A and B that are meshes.

The proof of Theorem 3 follows directly from the following two lemmas.

Lemma 5.1. *$H_{p,k}^j$ can be pebbled with $p+2kj+1$ black-white pebbles.*

Proof. Analogous to the proof of Lemma 3.2. ∎

Lemma 5.2. *Suppose $1 \le j \le p$ and $1 \le m \le p/k+1$. Consider any configuration of black pebbles on $H_{p,k}^j(m)$ such that column c is pebble-free. Let v be the vertex belonging to the last row of $H_{p,k}^j(m)$ and to column c.*
Then, any black pebble strategy to pebble v requires at least $(j-1)p+(m-1)k$ pebbles.

Proof. As before, we prove the claim by induction on j $(1 \le j \le p)$ and m $(1 \le m \le p/k+1)$.

Basis: ($j=1$ and $m=1$) The claim follows trivially.

First inductive step: (see figure 5) For integers r and s, assume that the claim holds for $1 < r < j \le p$ and for $r = j \le p$ and $1 \le s \le m \le p/k$. We will prove that the lemma also holds for $r=j$ and $s=m+1$.

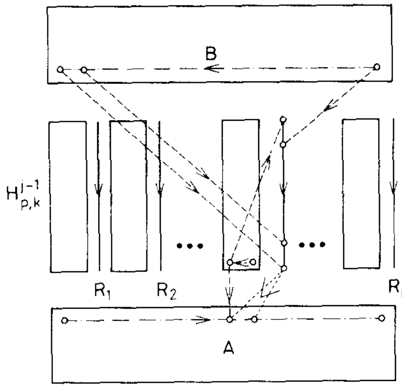


Fig. 5. The Block $H^j_{p,k}$

We define the boxes $Box(m+1, i)$ $(1 \leq i \leq k)$ as the set of all vertices that either belong to the last block of $H^j_{p,k}(m+1)$ and to a column d $((i-1)(p/k) \leq d \leq i(p/k))$ or to the components R^h_i $(1 \leq h \leq j)$.

Let us call the rows $4p^3 - 2p^2 + 1$ through $4p^3$ of A "witness" rows. These rows form p^2 pairs. We call a vertex v the "goal" vertex. Observe that the row number of the "goal" vertex is greater than the row number of any of the "witness" rows.

As before, we carry out a case analysis. Let $v_0(c)$ be the vertex that belongs to the first row of $H^j_{p,k}(m+1)$ and to column c. Also let $v_1(c)$ be the vertex belonging to the last row of the mth block of $H^j_{p,k}(m+1)$ and to column c. We consider a time interval I that starts (resp. ends) when a pebble is placed on $v_0(c)$ (resp. $v_1(c)$) for the first time.

Case 1: Throughout interval I, there is always a pebble in each of the boxes $Box(m+1, i)$ $(1 \leq i \leq k)$.

With our induction hypothesis we infer that $(j-1)p + (m-1)k + k$ pebbles are required (as claimed in the Lemma).

Case 2: Case 1 is false.

There is a time step (say t) in the interval I in which a box (say $Box(m+1, e)$) is empty. Since there are p^2 pairs of "witness" rows, at any time step there must be a pair of pebble-free rows. (Otherwise the lemma follows immediately.) Also there are p^2 pairs of rows in B. A similar argument shows that at any time step a pebble-free pair of rows in B exists. Therefore, at any time step t we can form a pebble-free path $q = (q_1, q_2, q_3, q_4)$ (see figure 5) traversing a pebble-free path (q_1, q_2). The path (q_2, q_3) will intersect the first (resp. last) row of $H^{j-1}_{p,k}$ of the $(m+1)$ block at a vertex q_a (resp. q_b). Finally path q returns back to the column containing the "goal" vertex through a pebble-free "witness" row $(q_3 q_4)$. Notice that the path q has to be pebbled. Also a vertex in q can be pebbled only after interval I.

Let J be the time interval that starts when a pebble is placed on q_2 and ends when a pebble is placed on q_3. Without loss of generality we can assume that the box e is not pebble-free at any time step in the interval J. (Otherwise, there is another time step in the interval J satisfying this property). There is subinterval J' of J that starts (resp. ends) when a pebble is placed on q_a (resp. q_b) for the first time in the interval J.

Let H be the induced subgraph of $H^j_{p,k}(m+1)$ obtained by removing the components A, R^j_i $(1 \le i \le k)$ and $H^{j-1}_{p,k}$ from the $(m+1)$th block.

Case 2.1: Throughout interval J', each column in the induced subgraph H possesses at least one pebble.

If $j = 1$, then $q_a = q_b$ and a pebble is placed on q_a. This gives a total of $p+1$ pebbles on $H^1_{p,k}(k)$. Therefore we are done.

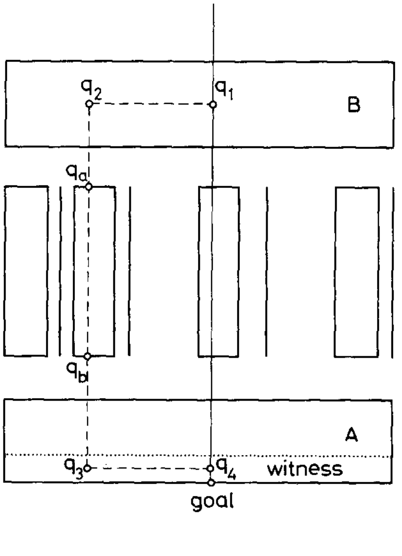
If $j > 1$, then we are also done because we can apply the induction hypothesis (within interval J') for $r = j-1$ and $s = p/k+1$.

Case 2.2: Case 2.1 is false. A column f becomes pebble-free at some instance in the interval J'. Notice that each vertex in the last row of B is connected to a vertex in R^j_e. Therefore, the column f (restricted to $H^j_{p,k}(m)$) has to be repebbled. Let the lowest row number of a "witness" row be w. Convert the witness rows $w, w+1, \ldots, w+p^2-1$ back to plain rows. Now, we call the rows $w-p^2, w-p^2+1, \ldots, w-1$ "witness" rows for the next iteration. Reset the "goal" vertex to q_3. Again observe that the pebble-free column restricted to the component A and containing the goal vertex has to be pebbled. This pebbling can only be started after pebbling the pebble-free column f restricted to the component H.

We can repeat the case analysis from the beginning by considering are pebble-free column f instead of column c.

Observe that we repeat the case analysis not more than $k+1$ times. (When we repeat the case analysis for the $k+1$st time each box will possess a pebble. Therefore, case 1 will be satisfied.)

Second inductive step: Assume that the claim holds for $r = j < p$ and $s = p/k+1$. We will prove that the claim also holds for $r = j+1 \le p$ and $s = 1$.



Since pebbling of the first block of $H^{j+1}_{p,k}$ implies pebbling of an isomorphic copy of $H^{j}_{p,k}$, by induction hypothesis at least $(j-1)p+p = jp$ black pebbles are required. ∎

Acknowledgements. We thank Piotr Berman and Martin Furer for many helpful comments.

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